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KETMA-KET KASR DIFFERENSIAL OPERATORLI TENGLAMALAR UCHUN APRIOR BAHOLAR VA YECHIMNING TURG'UNLIGI

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Annotatsiya. Ushbu tezisda ishda ketma-ket Riman–Liuvill kasr hosilali $D^{\alpha}D^{\beta}$ $(1 < \alpha + \beta \leq 2)$ nochiziqli tenglama uchun bir jinsli bo'lmagan chegaraviy masala yechimining aprior baholari va turg'unligi o'rganiladi. Grin funksiyasining aniq konstantasi asosida global va nuqtaviy aprior baholar olindi. Yechimning chegaraviy qiymat va o'ng tomonga Lipshits-uzluksiz bog'liqligi hamda Ulam–Hyers ma'nosida turg'unligi isbotlandi. Turg'unlik konstantasi $C = \Lambda/(1 - L\Lambda)$ oshkor ko'rinishda topildi. Sonli tajriba bahoning aniqligini tasdiqladi: o'zgarmas g'alayonda haqiqiy chetlanish nazariy bahoning 86 % ini tashkil etadi.

Kalit so'zlar: ketma-ket kasr hosila, aprior baho, uzluksiz bog'liqlik, Ulam–Hyers turg'unligi, Grin funksiyasi.

Аннотация. Для неоднородной краевой задачи с последовательной производной Римана–Лиувилля $D^{\alpha}D^{\beta}$ получены глобальные и поточечные априорные оценки, доказаны непрерывная зависимость решения от данных и устойчивость по Уламу–Хайерсу с явной константой $C = \Lambda/(1 - L\Lambda)$. Численный эксперимент подтверждает точность оценок.

Ключевые слова: последовательная дробная производная, априорная оценка, устойчивость по Уламу–Хайерсу.

Abstract. For a nonhomogeneous boundary value problem with the sequential Riemann–Liouville derivative $D^{\alpha}D^{\beta}$, global and pointwise a priori estimates are derived, Lipschitz dependence of the solution on the data and Ulam–Hyers stability with the explicit constant $C = \Lambda/(1 - L\Lambda)$ are proved. A numerical experiment confirms the sharpness of the estimates.

Keywords: sequential fractional derivative, a priori estimate, continuous dependence, Ulam–Hyers stability.

1. Kirish

Masala korrektligining Adamar ma'nosidagi uchinchi sharti — yechimning berilganlarga uzluksiz bog'liqligi — amaliy modellar uchun hal qiluvchi ahamiyatga ega. Chegaraviy qiymatlar va tashqi ta'sirlar o'lchash xatoliklari bilan beriladi, shuning uchun yechimning ularga sezgirligini oldindan baholash kerak. Bunday baholar aprior baholar deyiladi va ular sonli usullarning yaqinlashishini isbotlashda ham asosiy vosita hisoblanadi [1, 2].

Kasr tartibli tenglamalar uchun aprior baholar A.A. Alixanov [3] tomonidan energetik usul bilan olingan. Ulam [4] va Hyers [5] tomonidan funksional tenglamalar uchun kiritilgan turg‘unlik tushunchasi I.A. Rus [6] ishlarida differensial tenglamalarga ko‘chirildi. J. Wang va hammualliflar [7] esa uni Kaputo kasr hosilali tenglamalar uchun o‘rgandilar. Biroq ko‘pchilik ishlarda turg‘unlik konstantasi qo‘pol baholar orqali beriladi. Ketma-ket Riman–Liuvill operatorli chegaraviy masalalar uchun esa aniq konstantali natijalar deyarli yo‘q.

Avvalgi ishimizda [8] ketma-ket operatorli masala Grin funksiyasining aniq integral konstantasi $\Lambda(\gamma)$ topilgan edi. **Ushbu ishning maqsadi** — shu konstanta asosida bir jinsli bo‘lmagan masala uchun aprior baholarni olish, yechimning berilganlarga uzluksiz bog‘liqligini va Ulam–Hyers turg‘unligini aniq konstantalar bilan isbotlash hamda ularning aniqligini sonli tekshirish.

2. Materiallar va usullar

Tadqiqotda Grin funksiyasi usuli, Banax qisqartiruvchi akslantirishlar prinsipi va Gronuoll tipidagi baholash texnikasi qo‘llandi. Sonli tekshiruv uchun [8] dagi ikkinchi tartibli kvadratura sxemasidan ($N = 400$) foydalanildi.

1. Masalaning qo‘yilishi

$0 < \alpha, \beta \le 1$, $\gamma = \alpha + \beta \in (1,2]$ bo‘lsin. Bir jinsli bo‘lmagan masala qaraladi:

$$D_{0+}^{\alpha}\left(D_{0+}^{\beta}u\right)(t) = f\big(t,u(t)\big), \quad 0 < t < 1; \qquad \lim_{t\to 0+} t^{1-\beta}u(t) = 0, \quad u(1) = A. \qquad (1)$$

Bu yerda $A \in \mathbb{R}$, $f\colon [0,1]\times\mathbb{R}\to\mathbb{R}$ uzluksiz va x bo‘yicha L konstantali Lipshits shartini qanoatlantiradi. (1) masalani $P(A,f)$ deb belgilaymiz.

2. Integral ifoda

[8, Lemma 1] ga ko‘ra (1) masala quyidagi integral tenglamaga ekvivalent:

$$u(t) = A\,t^{\gamma-1} + \int_0^1 G\,(t,s)\,f\big(s,u(s)\big)\,ds, \qquad (2)$$

bu yerda G Grin funksiyasi uchun $G \le 0$ va quyidagi aniq tengliklar o‘rinli [8]:

$$\int_0^1 |G(t,s)|\,ds = \frac{t^{\gamma-1}(1-t)}{\Gamma(\gamma+1)} =: \psi(t), \qquad \Lambda = \max_{[0,1]}\psi = \frac{(\gamma-1)^{\gamma-1}}{\gamma^{\gamma}\,\Gamma(\gamma+1)}. \qquad (3)$$

3. Turg‘unlik ta’riflari

Ta’rif 1. $\varepsilon > 0$ bo‘lsin. $v \in C[0,1]$ funksiya (1) masalaning ε-yechimi deyiladi, agar u (1) dagi chegaraviy shartlarni qanoatlantirsa va

$$|D_{0+}^{\alpha}D_{0+}^{\beta}v(t) - f\big(t,v(t)\big)| \le \varepsilon, \qquad t \in (0,1). \qquad (4)$$

Ta’rif 2. (1) masala Ulam–Hyers ma’nosida turg‘un deyiladi, agar shunday $C > 0$ mavjud bo‘lsaki, har qanday $\varepsilon > 0$ va har qanday ε-yechim v uchun (1) ning $\| u - v \| \le C\varepsilon$ shartni qanoatlantiruvchi aniq yechimi u topilsa.

Keyingi barcha baholarda $L\Lambda < 1$ deb faraz qilinadi va quyidagi belgilash kiritiladi:

$$K := \frac{1}{1 - L\Lambda}. \qquad (5)$$

3. Natijalar

1. Aprior baholar

Teorema 1. $L\Lambda < 1$ va $M_0 := \max|f(t,0)|$ bo‘lsin. U holda $P(A,f)$ masalaning har bir yechimi uchun global va nuqtaviy baholar o‘rinli:

$$\| u \| \leq K(|A| + \Lambda M_0), \qquad (6)$$

$$|u(t)| \leq |A|\, t^{\gamma-1} + K(L|A| + M_0)\, \psi(t), \qquad t \in [0,1]. \qquad (7)$$

Isbot. (2) va $|f(s,u)| \leq L|u| + M_0$ dan $|u(t)| \leq |A|t^{\gamma-1} + \psi(t)(L \| u \| + M_0)$. Maksimum olib, $\| u \| \leq |A| + \Lambda(L \| u \| + M_0)$ ga ega bo‘lamiz, bu (6) ni beradi. (6) ni qayta qo‘ysak, $L \| u \| + M_0 \leq K(L|A| + M_0)$, bu esa (7) ni beradi.

2. Berilganlarga uzluksiz bog‘liqlik

Teorema 2. u_i — $P(A_i, f_i)$ masalalarning yechimlari ($i = 1,2$) bo‘lsin. f_1 L konstantali Lipshits shartini qanoatlantirsin, $L\Lambda < 1$ va $\delta := \sup|f_1(t,x) - f_2(t,x)|$. U holda:

$$\| u_1 - u_2 \| \leq K(|A_1 - A_2| + \Lambda\, \delta). \qquad (8)$$

Isbot. $w = u_1 - u_2$ ayirma uchun (2) dan

$$w(t) = (A_1 - A_2)\, t^{\gamma-1} + \int_0^1 G\,(t,s)[f_1(s,u_1) - f_1(s,u_2) + f_1(s,u_2) - f_2(s,u_2)]ds.$$

Kvadrat qavs ichidagi ifoda moduli $L|w| + \delta$ dan oshmaydi. Shuning uchun $\| w \| \leq |A_1 - A_2| + \Lambda(L \| w \| + \delta)$, bundan (8) kelib chiqadi.

Natija 1. $P(A,f)$ masalaning yechimi yagona va A, f berilganlarga Lipshits-uzluksiz bog‘liq. Demak, masala $C[0,1]$ fazoda Adamar ma’nosida korrekt.

3. Ulam–Hyers turg‘unligi

Teorema 3. $L\Lambda < 1$ bo‘lsin. U holda (1) masala Ulam–Hyers ma’nosida turg‘un. Har bir ε-yechim v uchun aniq yechim u mavjudki,

$$\| u - v \| \leq C_{UH}\, \varepsilon, \qquad C_{UH} = \frac{\Lambda}{1 - L\Lambda}; \qquad |u(t) - v(t)| \leq K\varepsilon\, \psi(t). \qquad (9)$$

Isbot. $h(t) := D^{\alpha}D^{\beta}v(t) - f\big(t, v(t)\big)$ deylik, $|h| \leq \varepsilon$. U holda v funksiya $P(A, f + h)$ masalaning yechimi, u esa $P(A,f)$ ning yagona yechimi. (8) ni $A_1 = A_2$, $\delta = \varepsilon$ bilan qo‘llasak, birinchi baho kelib chiqadi. Nuqtaviy baho $w = u - v$ uchun $|w(t)| \leq \psi(t)(L \| w \| + \varepsilon) \leq \psi(t)(LC_{UH} + 1)\varepsilon = K\varepsilon\psi(t)$ dan kelib chiqadi.

Izoh 1 (aniqlik). $L = 0$ bo‘lganda (f u ga bog‘liq emas) va $h \equiv \varepsilon$ da $|w(t)| = \varepsilon\psi(t)$, ya’ni (9) baho $t^* = (\gamma - 1)/\gamma$ nuqtada tenglikka aylanadi. Demak, C_{UH} konstantani umumiy holda yaxshilab bo‘lmaydi.

4. Sonli tekshiruv

[8] dagi masala qaraladi: $\alpha = 0{,}6$, $\beta = 0{,}8$, $f(t,u) = \sin u/(4+t) + t^2 - 1/2$, $A = 0$. Bu yerda $L = 0{,}25$, $\Lambda = 0{,}34839$, $K = 1{,}09541$ va $C_{UH} = 0{,}38163$. Ikki xil g'alayon berildi: o'zgarmas $h \equiv \varepsilon$ va tebranuvchi $h = \varepsilon\sin 5\pi t$.

1-jadval

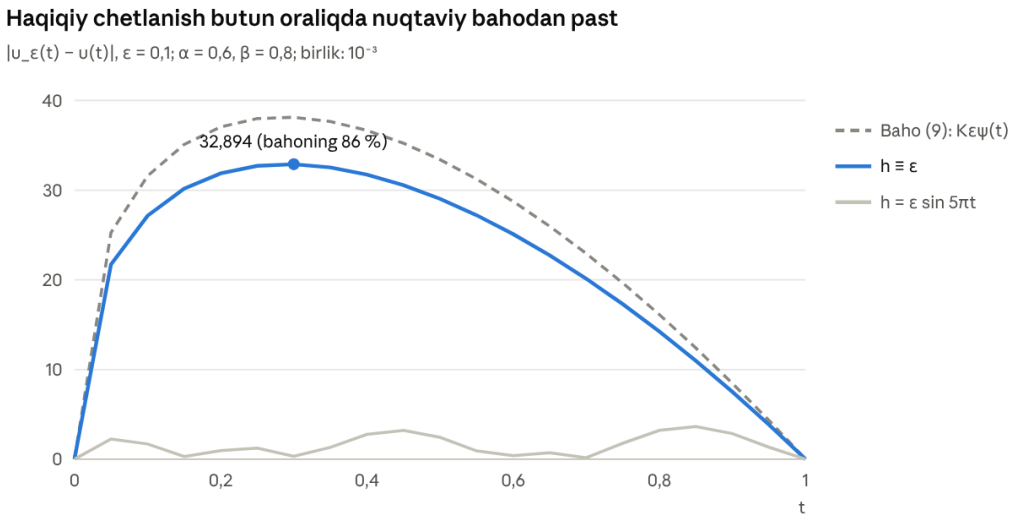
O'ng tomon g'alayonida chetlanish va (9) baho

ε	$\| u_\varepsilon - u \|$, $h \equiv \varepsilon$	$\| u_\varepsilon - u \|$, $h = \varepsilon\sin 5\pi t$	$C_{UH}\varepsilon$ **(baho)**	**Nisbat,** $h \equiv \varepsilon$	**Nisbat, tebranuvchi**
0,1	$3{,}2905 \cdot 10^{-2}$	$3{,}6598 \cdot 10^{-3}$	$3{,}8163 \cdot 10^{-2}$	0,862	0,096
0,05	$1{,}6453 \cdot 10^{-2}$	$1{,}8299 \cdot 10^{-3}$	$1{,}9081 \cdot 10^{-2}$	0,862	0,096
0,01	$3{,}2906 \cdot 10^{-3}$	$3{,}6598 \cdot 10^{-4}$	$3{,}8163 \cdot 10^{-3}$	0,862	0,096
0,001	$3{,}2906 \cdot 10^{-4}$	$3{,}6598 \cdot 10^{-5}$	$3{,}8163 \cdot 10^{-4}$	0,862	0,096

Chetlanish ε ga chiziqli bog'liq va barcha hollarda bahodan kichik. O'zgarmas g'alayonda u bahoning 86,2 % iga yetadi, ya'ni baho amalda aniq. Tebranuvchi g'alayonda Grin integrali musbat va manfiy qismlarni o'zaro so'ndiradi, shuning uchun chetlanish 10 marta kichik.

Chegaraviy qiymat $\Delta A \in \{10^{-1}, 10^{-2}, 10^{-3}\}$ ga g'alayonlanganda chetlanish aynan ΔA ga teng (maksimum $t = 1$ da), (8) baho esa $K\Delta A = 1{,}0954\,\Delta A$, ya'ni 9,5 % zaxira bilan qoplaydi.

1-rasmda $\varepsilon = 0{,}1$ uchun chetlanishning t bo'yicha taqsimoti (9) nuqtaviy baho bilan solishtirilgan.



1-rasm. Chetlanishning nuqtaviy taqsimoti va (9) baho ($\varepsilon = 0,1$): o‘zgarmas g‘alayonda profil $K\varepsilon\psi(t)$ bahoning shaklini takrorlaydi, maksimum $t^* = 0,286$ yaqinida

4. Muhokama

Olingan baholarning asosiy afzalligi — konstantalarning oshkor va aniq ekanligi. 2-jadvalda ular adabiyotda keng tarqalgan qo‘pol yondashuv bilan solishtirilgan; qo‘pol yondashuvda Λ o‘rniga $2/\Gamma(\gamma+1)$ olinadi [7].

2-jadval. Turg‘unlik konstantalarini solishtirish ($L = 0,25$)

γ	Λ, aniq	C_{UH}, aniq	$2/\Gamma(\gamma+1)$, qo‘pol	C_{UH}, qo‘pol	Yaxshilanish, marta
1,2	0,52855	0,60902	1,81521	3,32335	5,46
1,4	0,34839	0,38163	1,61009	2,69480	7,06
1,6	0,24270	0,25838	1,39897	2,15141	8,33
1,8	0,17321	0,18105	1,19297	1,69997	9,39
2,0	0,12500	0,12903	1,00000	1,33333	10,33

Aniq konstanta turg‘unlik konstantasini 5–10 marta kamaytiradi. Bundan tashqari, qo‘pol yondashuvda $L \cdot 2/\Gamma(\gamma+1) < 1$ shart $L < 0,55$ ($\gamma = 1,2$) ni talab qiladi, aniq yondashuvda esa $L < 1,89$ yetarli.

(9) baho eng yomon holatga mo‘ljallangan: ishorasi o‘zgaruvchan xatoliklarda Grin funksiyasining silliqlovchi ta’siri tufayli chetlanish ancha kichik (1-jadval, 1-rasm). Cheklov sifatida natijalar global Lipshits shartiga tayanishini qayd etamiz; lokal Lipshits holida baholar (6) aprior baho bilan chegaralangan shar ichida o‘rinli bo‘ladi.

5. Xulosa

1. Ketma-ket Riman–Liuvill kasr hosilali bir jinsli bo‘lmagan chegaraviy masala uchun global (6) va nuqtaviy (7) aprior baholar olindi.

2. Yechimning chegaraviy qiymat va o‘ng tomonga Lipshits-uzluksiz bog‘liqligi (8) isbotlandi, ya’ni masala Adamar ma’nosida korrekt.
3. Masalaning Ulam–Hyers turg‘unligi $C_{UH} = \Lambda/(1 - L\Lambda)$ aniq konstanta bilan isbotlandi. $L = 0$ da bu konstantani yaxshilab bo‘lmaydi.
4. Sonli tajribada haqiqiy chetlanish nazariy bahoning 86 % iga yetdi. Aniq konstanta qo‘pol bahoga nisbatan 5–10 marta kichik turg‘unlik konstantasini beradi.

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